MATH3705 Tutorial 5

1. Solve $u_{xx} = \frac{1}{9}u_t$, subject to the boundary conditions u(0, t) = 2, u(2, t) = 4, and the initial condition u(x, 0) = 5.

   Solution:

   $$u(x,t) = (x+2) + \sum_{n=1}^{\infty} \frac{2}{n\pi}[3-(-1)^n] \sin\left(\frac{n\pi x}{2}\right) e^{-\left(\frac{3n\pi}{2}\right)^2 t}.$$

2. Solve $2u_{tt} = u_{xx}$, subject to the boundary conditions u(0, t) = u(3, t) = 0 and the initial conditions $u(x,0) = 0, u_t(x,0) = 3\sin(6\pi x) - 2sin(9\pi x)$.

   Solution:

   $$u(x,t) = \frac{1}{\sqrt{2}\pi} \sin(6\pi x) \sin\left(\frac{6\pi t}{\sqrt{2}}\right) - \frac{2\sqrt{2}}{9\pi} \sin(9\pi x) \sin\left(\frac{9\pi t}{\sqrt{2}}\right)$$

3. Find a polynomial solution $u(x,y) = ax + by + cxy + d$ of Laplace's equation within

   $$R = \{(x,y) : 0 < x < 1, 0 < y < 2\}$$

   with

   $$\text{BC}: \quad u(x,0) = 0, u(x,2) = 1 - x, u(0,y) = y/2, u(1,y) = 0.$$

   Solution:

   $$u(x,y) = y(1-x)/2.$$

4. Find the bounded solution of $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$ outside the circle $r = 3$, subject to the boundary condition $u(3,\theta) = 2\cos^2(\theta) - 4\sin^2(\theta)$.

   Solution:

   $$u(r,\theta) = -1 + \frac{27}{r^2}\cos(2\theta).$$